12.3 Oversampling of Analog-to-Digital Conversion

Oversampling of the analog signal has become popular in DSP industry to improve resolution of analog-to-digital conversion (ADC). Oversampling uses a sampling rate, which is much higher than the Nyquist rate. We can define an oversampling ratio as

\[
\frac{f_s}{2f_{\text{max}}} \gg 1. \quad (12.14)
\]

The benefits from an oversampling ADC include:

1. helping to design a simple analog anti-aliasing filter before ADC, and
2. reducing the ADC noise floor with possible noise shaping so that a low-resolution ADC can be used.

12.3.1 Oversampling and Analog-to-Digital Conversion Resolution

To begin with developing the relation between oversampling and ADC resolution, we first summarize the regular ADC and some useful definitions discussed in Chapter 2:

Quantization noise power = \( \sigma_q^2 = \frac{\Delta^2}{12} \) (12.15)

Quantization step = \( \Delta = \frac{A}{2^n} \) (12.16)

where

\( A = \) full range of the analog signal to be digitized
\( n = \) number of bits per sample (ADC resolution).

Substituting Equation (12.16) into Equation (12.15), we have:

Quantization noise power = \( \sigma_q^2 = \frac{A^2}{12} \times 2^{2n} \) (12.17)
The power spectral density of the quantization noise with an assumption of uniform probability distribution is shown in Figure 12-23. Note that this assumption is true for quantizing a uniformly distributed signal in a full range with a sufficiently long duration. It is not generally true in practice. See research papers by Lipshitz et al. (1992) and Maher (1992). However, using the assumption will guide us for some useful results for oversampling systems.

The quantization noise power is the area obtained from integrating the power spectral density function in the range of $-f_s/2$ to $f_s/2$. Now let us examine the oversampling ADC, where the sampling rate is much bigger than that of the regular ADC; that is, $f_s \gg 2f_{\text{max}}$. The scheme is shown in Figure 12-24.

As we can see, oversampling can reduce the level of noise power spectral density. After the decimation process with the decimation filter, only a portion of quantization noise power in the range from $-f_{\text{max}}$ to $f_{\text{max}}$ is kept in the DSP system. We call this an in-band frequency range. In Figure 12-24, the shaded area, which is the quantization noise power, is given by

\[
\text{quantization noise power} = \int_{-\infty}^{\infty} P(f) df = \frac{2f_{\text{max}}}{f_s} \times \sigma_q^2
\]

\[
= \frac{2f_{\text{max}}}{f_s} \times \frac{A^2}{12} \times 2^{-2m}.
\]
Assuming that the regular ADC shown in Figure 12-23 and the oversampling ADC shown in Figure 12-24 are equivalent, we set their quantization noise powers to be the same to obtain

\[(A^2/12) \times 2^{-2n} = ((2f_{\text{max}})/(f_s)) \times (A^2/12) \times 2^{-2m}. \quad (12.19)\]

Equation (12.19) leads to two useful equations for applications:

\[n = m + 0.5 \times \log_2 \left(\frac{f_s}{(2f_{\text{max}})}\right) \quad (12.20)\]

and

\[f_s = 2f_{\text{max}} \times 2^{2(n-m)}, \quad (12.21)\]

where

- \(f_s\) = sampling rate in the oversampling DSP system
- \(f_{\text{max}}\) = maximum frequency of the analog signal
- \(m\) = number of bits per sample in the oversampling DSP system
- \(n\) = number of bits per sample in the regular DSP system using the minimum sampling rate

From Equation (12.20) and given the number of bits \((m)\) used in the oversampling scheme, we can determine the number of bits per sample equivalent to the regular ADC. On the other hand, given the number of bits in the oversampling ADC, we can determine the required oversampling rate so that the oversampling ADC is equivalent to the regular ADC with the larger number of bits per sample \((n)\). Let us look at the following examples.

**Example 12.7.**
Given an oversampling audio DSP system with the following attributes, determine the oversampling rate to improve the ADC to 16-bit resolution:

- Maximum audio input frequency of 20 kHz and
- ADC resolution of 14 bits,

Solution: Based on the specifications, we have

\[f_{\text{max}} = 20 \text{ kHz}, \quad m = 14 \text{ bits}, \quad \text{and} \quad n = 16 \text{ bits}.\]

Using Equation (12.21) leads to

\[f_s = 2f_{\text{max}} \times 2^{2(n-m)} = 2 \times 20 \times 2^{2(16-14)} = 640 \text{ kHz}.\]

Since \(f_s/(2f_{\text{max}}) = 2^{1}\), we see that each doubling of the minimum sampling rate \((2f_{\text{max}} = 40 \text{ kHz})\) will increase the resolution by a half bit.

**Example 12.8.**
Given an oversampling audio DSP system with the following attributes, determine the equivalent ADC resolution:

- Maximum audio input frequency = 4 kHz
- ADC resolution = 8 bits
- Sampling rate = 80 MHz.

Solution: Since \(f_{\text{max}} = 4 \text{ kHz}, f_s = 80 \text{ kHz}, \) and \(m = 8 \text{ bits},\) applying Equation (12.20) yields
\[ n = m + 0.5 \times \log_2 \left( \frac{f_s}{2f_{\text{max}}} \right) = 8 + 0.5 \times \log_2 \left( \frac{80000 \text{ kHz}}{2 \times 4 \text{ kHz}} \right) \approx 15 \text{ bits}. \]

**Sigma-Delta Modulation Analog-to-Digital Conversion**

**12.3.2 Sigma-Delta Modulation Analog-to-Digital Conversion**

To further improve ADC resolution, *sigma-delta modulation* (SDM) ADC is used. The principles of the first-order SDM are described in Figure 12-25.

**Figure 12-25. Block diagram of SDM ADC.**

First, the analog signal is sampled to obtain the discrete-time signal \( x(n) \). This discrete-time signal is subtracted by the analog output from the \( m \)-bit DAC, converting the \( m \) bit oversampled digital signal \( y(n) \). Then the difference is sent to the discrete-time analog integrator, which is implemented by the switched-capacitor technique, for example. The output from the discrete-time analog integrator is converted using an \( m \)-bit ADC to produce the oversampled digital signal. Finally, the decimation filter removes outband quantization noise. Further decimation process can change the oversampling rate back to the desired sampling rate for the output digital signal \( w(m) \). To examine the SDM, we need to develop a DSP model for the discrete-time analog filter described in Figure 12-26.

**Figure 12-26. Illustration of discrete-time analog integrator.**

As shown in Figure 12-26, the input signal \( c(n) \) designates the amplitude at time instant \( n \), while the output \( d(n) \) is the area under the curve at time instant \( n \), which can be expressed as a sum of the area under the curve at time instant \( n - 1 \) and area increment:

\[ d(n) = d(n - 1) + \text{area incremental}. \quad (12.22) \]

Using the extrapolation method, we have

\[ d(n) = d(n - 1) + 1 \times c(n). \quad (12.23) \]

Applying the z-transform to Equation (12.23) leads to a transfer function of the discrete-time analog filter as

\[ H(z) = \frac{D(z)}{C(z)} = \frac{1}{1 - z^{-1}}. \quad (12.24) \]

Again, considering that the \( m \)-bit quantization requires one sample delay, we get the DSP model for the first-order SDM depicted in Figure 12-27, where \( y(n) \) is the oversampling data encoded by \( m \) bits.
each, and $e(n)$ represents quantization error.

![Figure 12-27. DSP model for first-order SDM ADC.](image)

The SDM DSP model represents a feedback control system. Applying the z-transform leads to

$$Y(z) = \frac{1}{1 - z^{-1}}(X(z)) - z^{-1}Y(z) + E(z). \quad (12.25)$$

After simple algebra, we have

$$Y(z) = \underbrace{X(z)}_{\text{Original digital signal transform}} + \underbrace{(1 - z^{-1})}_{\text{Highpass filter}} \cdot \underbrace{E(z)}_{\text{Quantization error transform}}. \quad (12.26)$$

In Equation (12.26), the indicated highpass filter pushes quantization noise to the high-frequency range, where later the quantization noise can be removed by the decimation filter. Thus we call this highpass filter $(1 - z^{-1})$ the noise shaping filter, illustrated in Figure 12-28.

![Figure 12-28. Noise shaping of quantization noise for SDM ADC.](image)

Shaped-in-band noise power after use of the decimation filter can be estimated by the solid area under the curve. We have
Using the Maclaurin series expansion and neglecting the higher-order term due to the small value of $\Omega_{\text{max}}$, we yield

$$1 - e^{-j\Omega} = 1 - (1 + (-j\Omega)/1! + (-j\Omega)^2/2! + \ldots) \approx j\Omega.$$ 

Applying this approximation to Equation (12.27) leads to

$$\text{Shaped-in-band noise power} \approx \int_{-\Omega_{\text{max}}}^{\Omega_{\text{max}}} \frac{\sigma_q^2}{2\pi} |j\Omega|^2 d\Omega = \frac{\sigma_q^2}{3\pi} \Omega_{\text{max}}^3.$$ 

After simple algebra, we have

$$\text{Shaped-in-band noise power} \approx \left(\frac{\pi^2 \sigma_q^2}{3}\right) \times \left(\frac{(2f_{\text{max}})/f_s)^3}{(A^2 2^{-2m})/12}\times \left(2f_{\text{max}})/f_s)^3 = \left(\frac{A^2}{12}\right) \times 2^{-2n}\right.$$ (12.29)

If we let the shaped-in-band noise power equal the quantization noise power from the regular ADC using a minimum sampling rate, we have

$$\left(\frac{\pi^2}{3}\right) \times \left(\frac{(A^2 2^{-2m})/12}\times \left(2f_{\text{max}})/f_s)^3 = \left(\frac{A^2}{12}\right) \times 2^{-2n}\right.$$ (12.30)

We modify Equation (12.30) into the following useful formats for applications:

$$n = m + 1.5 \times \log_2 \left(\frac{f_s}{2f_{\text{max}}}\right) - 0.86 \ (12.31)$$

$$(f_s/(2f_{\text{max}}))^3 = \left(\frac{\pi^2}{3}\right) \times 2^{2(n-m)} \ (12.32)$$

**Examples**

**Example 12.9.**

Given the following DSP system specifications, determine the equivalent ADC resolution:

- Oversampling rate system
- First-order SDM with 2-bit ADC
- Sampling rate = 4 MHz
- Maximum audio input frequency = 4 kHz.

Solution: Since $m = 2$ bits, and

$$f_s/(2f_{\text{max}}) = 4000 \text{ kHz}/(2 \times 4 \text{ kHz}) = 500.$$ 

we calculate

$$n = m + 1.5 \times \log_2 \left(\frac{f_s}{2f_{\text{max}}}\right) - 0.86 = 2 + 1.5 \times \log_2 (500) - 0.86 \approx 15 \text{ bits}.$$ 

We can also extend the first-order SDM DSP model to the second-order SDM DSP model by cascading one section of the first-order discrete-time analog filter, as depicted in Figure 12-29.
Figure 12-29. DSP model for the second-order SDM ADC.

Similarly to the first-order SDM DSP model, applying the z-transform leads to the following relationship:

\[ Y(z) = \frac{X(z)}{1 - z^{-1}} + \left(1 - z^{-1}\right)^2 E(z). \quad (12.33) \]

Notice that the noise shaping filter becomes a second-order highpass filter; hence, the more quantization noise is pushed to the high-frequency range, the better ADC resolution is expected to be. In a similar analysis to the first-order SDM, we get the following useful formulas:

\[ n = m + 2.5 \times \log_2 \left(\frac{f_s}{2f_{\text{max}}}\right) - 2.14 \quad (12.34) \]

\[ \left(\frac{f_s}{2f_{\text{max}}}\right)^{\frac{5}{2}} = \left(\pi^5/5\right) \times 2^{2n - m}. \quad (12.35) \]

In general, the Kth-order SDM DSP model and ADC resolution formulas are given as:

\[ Y(z) = \frac{X(z)}{1 - z^{-1}} + \left(1 - z^{-1}\right)^K E(z) \quad (12.36) \]

\[ n = m + 0.5 \times (2K + 1) \times \log_2 \left(\frac{f_s}{2f_{\text{max}}}\right) - 0.5 \times \log_2 \left(\pi^{2K}/(2K + 1)\right) \quad (12.37) \]

\[ \left(\frac{f_s}{2f_{\text{max}}}\right)^{2K + 1} = \left(\pi^{2K}/(2K + 1)\right) \times 2^{2(n - m)}. \quad (12.38) \]

**Example 12.10.**
Given the oversampling rate DSP system with the following specifications, determine the effective ADC resolution:

- Second-order SDM = 1-bit ADC
- Sampling rate = 1 MHz
- Maximum audio input frequency = 4 kHz.

Solution:

\[ n = 1 + 2.5 \times \log_2 \left(\frac{1000\ kHz}{24\ kHz}\right) - 2.14 \approx 16 \text{ bits}. \]
Next, we review the application of the oversampling ADC used in industry. Figure 12-30 illustrates a function diagram for the MAX1402 low-power, multichannel oversampling sigma-delta analog-to-digital converter used in industry. It applies a sigma-delta modulator with a digital decimation filter to achieve 16-bit accuracy. The device offers three fully differential input channels, which can be independently programmed. It can also be configured as five pseudo-differential input channels. It comprises two chopper buffer amplifiers and a programmable gain amplifier, a DAC unit with predicted input subtracted from the analog input to acquire the differential signal, and a second-order switched-capacitor sigma-delta modulator.

![Figure 12-30. Functional diagram for the sigma-delta ADC.](Click to enlarge)

The chip produces a 1-bit data stream, which will be filtered by the integrated digital filter to complete ADC. The digital filter's user-selectable decimation factor offers flexibility for conversion resolution to be reduced in exchange for a higher data rate, or vice versa. The integrated digital lowpass filter is first-order or third-order Sinc infinite impulse response. Such a filter offers notches corresponding to its output data rate and its frequency harmonics, so it can effectively reduce the developed image noises in the frequency domain. (The Sinc filter is beyond the scope of our discussion.) The MAX1402 can provide 16-bit accuracy at 480 samples per second and 12-bit accuracy at 4,800 samples per second. The chip finds wide application in sensors and instrumentation. Its detailed features can be found in the MAX1402 data sheet (Maxim Integrated Products, 2007).

### Application Example: CD Player

**12.4 Application Example: CD Player**

Figure 12-31 illustrates a CD playback system, also described earlier in this chapter. A laser optically scans the tracks on a CD to produce a digital signal. The digital signal is then demodulated, and parity bits are used to detect bit errors due to manufacturing defects, dust, and so on and to correct them. The demodulated signal is again oversampled by a factor of 4 and hence the sampling rate is increased to 176.4 kHz for each channel. Each digital sample then passes through a 14-bit DAC, which produces the sample-and-hold voltage signals that pass the anti-image lowpass filter. The output from each analog filter is fed to its corresponding loudspeaker. Oversampling relaxes the design requirements of the analog anti-image lowpass filter, which is used to smooth out the voltage steps.
The earliest system used a third-order Bessel filter with a 3 dB passband at 30 kHz. Notice that the first-order sigma-delta modulation (first-order SDM) is added to the 14-bit DAC unit to further improve the 14-bit DAC to 16-bit DAC.

Let us examine the single-channel DSP portion shown in Figure 12-32.

The spectral plots for the oversampled and interpolated signal $x(n)$, the 14-bit SDM output $y(n)$, and the final analog output audio signal are given in Figure 12-33.
As we can see in plot (a) in the figure, the quantization noise is uniformly distributed, and only in-band quantization noise (0 to 22.05 kHz) is expected. Again, 14 bits for each sample are kept after oversampling. Without using the first-order SDM, we expect the effective ADC resolution due to oversampling to be

\[ n = 14 + 0.5 \times \log_2 \left( \frac{176.4}{44.1} \right) = 15 \text{ bits}, \]

which is fewer than 16 bits. To improve quality further, the first-order SDM is used. The in-band quantization noise is then shaped. The first-SDM pushes quantization noise to the high-frequency range, as illustrated in plot (b) in Figure 12-33. The effective ADC resolution now becomes

\[ n = 14 + 1.5 \times \log_2 \left( \frac{176.4}{44.1} \right) - 0.86 \approx 16 \text{ bits}. \]

Hence, 16-bit ADC audio quality is preserved. On the other hand, from plot (c) in Figure 12-33, the audio occupies a frequency range up to 22.05 kHz, while the DSP Nyquist limit is 88.2, so the low-order analog anti-image filter can satisfy the design requirement.

*Part 4 examines the undersampling of bandpass signals.*

Note that wavelet transform and subband coding are also in the area of multirate signal processing. We do not pursue these subjects in this book. The reader can find useful fundamental information in Akansu and Haddad (1992), Stearns (2003), Van der Vegte (2002), and Vetterli and Kovacevic (1995).

Related articles:

- Design: [ADCs for DSP, part 1](#)
- Product: [Low noise 16-bit delta sigma A/D converter upgrades system accuracy](#)
- Paper: [Benefits of Sigma Delta ADCs](#)