The single-stage single-pole approximation

Dennis Feucht - May 13, 2013

Can long-standing methods in circuit analysis be in need of some refinement? When it comes to circuit dynamics, many engineers are quick to invoke their computer circuit simulation programs and let SPICE give numerically accurate answers. That’s great if you already know what the circuit is, but what if you are designing? Then some good approximating knowledge can guide choice of components. And one well-known bandwidth approximation for single-stage amplifiers is in need of refinement because it is conceptually misleading and too often numerically inaccurate.

It is commonly found throughout the engineering circuits literature that the input-loop dynamic response of a single BJT or FET stage can be approximated by a single equivalent base capacitance, $C_{\text{eq}}$, that forms a time constant with the base-node resistance, $R_b = R_E \parallel (\beta_0 +1) \cdot (r_e + R_E)$

where $R_b$ is the external base resistance in parallel with the input resistance of the base port, on the BJT side of the port. The capacitance is a combination of the Miller capacitance of the collector-base BJT (or corresponding FET) junction capacitance, $C_c$ or $C_e$, and the BJT emitter-base capacitance, $C_n$ or $C_v$. Some textbook examples demonstrate the method.

Beginning with my first choice of active-circuits textbook, *Electronics: BJTs, FETs, and Microcircuits*, by Brooklyn Tech professor E. James Angelo, Jr. (McGraw-Hill, 1969), section 14.5 (pp. 426 - 430), the analysis assumes $R_E = 0 \Omega$ (as do textbooks typically, for simplicity). Then $C_n$ is at the base node to ground. By applying Miller’s theorem, in parallel with it is (beginning with Angelo’s notation and translating it to mine),

$$(1 + g_m \cdot R_L) \cdot C_n - (1 + \alpha_0 \cdot \frac{r_e}{R_E}) \cdot C_e - (1 + K_v) \cdot C_v$$

where the incremental emitter resistance,

$$r_e = \frac{V_T}{|I_E|} \approx \frac{26 \text{ mV}}{|I_E|}$$

is related to transconductance as

$$\frac{1}{g_m} - r_m = \frac{r_e}{\alpha_0}$$

and where $\alpha_0$ is the quasistatic (frequency-independent or “low-frequency ac”) $\alpha$. Combining the
capacitances,
\[ C_b - C_e + (1 + K_v) \cdot C_c , \quad K_v = \frac{R_f}{r_e} \]

\( C_b \) forms a time constant, \( R_f \cdot C_b \) at the base node. This time constant determines the single pole that is the approximation of the input-loop bandwidth. (At the collector, another pole is added based on the time constant \( R_e \cdot C_L \) where \( C_i \) is the collector node capacitance and does not generally include \( C_c \).)

In a more rigorous and notationally-challenging circuits textbook of the same year, *Electronic Principles: Physics, Models, and Circuits*, by MIT professors Paul E. Gray (who is different from another circuits textbook author on the opposite coast, Paul R. Gray) and Campbell Searle (Wiley, 1969), cover the topic in section 14.5.2, “The One-Pole Approximation” (pp. 499 - 503). (Note: not all circuits textbooks cover this topic in section 14.5.) They also assume \( R_b = 0 \ \Omega \) and refer to the pole polynomial in the circuit transfer function, which (upon normalization, changing conductances to resistances, and translating symbols, to reduce “equation entropy”) is equivalent to

\[ s^2 \cdot [(R_b \cdot C_e) \cdot (R_L \cdot C_c)] + s \cdot [R_b \cdot C_s + R_L \cdot C_c + R_b \cdot (1 + K_v) \cdot C_c] + 1 , \quad K_v = \frac{R_f}{r_e} \]

They conclude that the model is valid only for frequencies much less than \( f_T \), for which the value of the \( s^2 \) term is negligible and is neglected. (Such an approximation is too limiting to apply high-frequency theory, which covers circuit behavior between bandwidth and \( f_T \).) This neglects a pole that they say is always greater than \( f_T \). The resulting one-pole polynomial is

\[ s \cdot [R_b \cdot C_s + R_L \cdot C_c + R_b \cdot (1 + K_v) \cdot C_c] + 1 \]

However, instead of simplifying the coefficient by collecting \( R_b \) terms as above, they further “simplify” it to result in an equivalent base capacitance of

\[ C_i - C_e + C_m [1 + (\xi_m + C_i) \cdot R_L] \Rightarrow C_i - C_e + (1 + \frac{R_e}{R_b}) \cdot C_c \]

This is not quite the same approximation as that made by Angelo in that it includes \( R_b' = R_b + r_b' \) in \( K_v \). However, the same basic procedure underlies the method.

A third and final example, in *Electronic Amplifier Circuits: Theory and Design*, by Joseph Petit and Malcolm McWhorter of Stanford U. (McGraw-Hill, 1961), in section 3.9, “Determination of the Amplifier High-frequency Response” (pp. 74 - 81), uses two-port hybrid parameters to model the BJT, but nevertheless results in the same equivalent \( C_b \) as Angelo; in their notation,

\[ C_{eq} = \frac{1}{\omega_T \cdot r_e \cdot (1 + \alpha_0 R_L / r_e') \cdot C_s} = \frac{1}{\omega_T \cdot r_e} \cdot (1 + \omega_T \cdot C_c \cdot R_L) \]

which translates to

\[ C_b - \frac{\tau}{r_e} + (1 + \alpha_0 \cdot \frac{R_e}{r_e}) \cdot C_c \approx C_b + (1 + K_v) \cdot C_c \]

Their approximation reduces (unnecessarily) the second term to \((R_e/r_e)\cdot C_c\). (To be exact, \( \tau / r_e = C_j \alpha_0 \).)

The same method can also be found in my book, *Designing High-Performance Amplifiers*, D. Feucht
OCTCs and Nodal Time Constants

It has been shown in a previous article of this series that the open-circuit time constants (OCTCs) of reactances (whereby the time constant is formed by a given capacitance with the circuit resistance across its terminals) are added in the linear term (the $s$ coefficient) of the pole polynomial. When $R_E$ is placed back into the single-stage BJT amplifier (so that $R_I$, $R_B$, and $R_E$ are all there), the general single-stage amplifier results for which $R_{br}$, of the OCTC of $C_e$, is not $R_b$ in general but only when $R_E = 0 \, \Omega$.

The OCTC linear pole coefficient for the textbook BJT stage (with $R_E = 0 \, \Omega$) is, as all three authors derived it,

$$R_{se} \cdot C_e + (R_L + R_b \cdot (1 + K_v)) \cdot C_c = [R_{se} \cdot C_e + R_b \cdot C_c \cdot (1 + K_v)] + R_L \cdot C_c$$

where $R_b$ is the base node resistance (to ground). The OCTC bandwidth formula requires the TC groupings according to capacitance as in the left-side expression:

$$\tau_c = \tau_{se} = R_{se} \cdot C_e$$
$$\tau_c = \tau_{cc} + \tau_{cb} = R_I \cdot C_c + R_b \cdot [(1 + K_v) \cdot C_c]$$

The terms of $\tau_c$ are designated as time constant components according to node. When the OCTCs are rearranged and separated by node,

$$\tau_b = R_{se} \cdot C_e + R_b \cdot [(1 + K_v) \cdot C_c] = \tau_{se} + \tau_{cb}$$
$$\tau_{cc} = R_I \cdot C_c$$

The $\tau_b$ time constant is what is commonly used in bandwidth approximation though it mixes OCTC terms and is not generally valid. The difference in $\tau_{bw}$ that results from the OCTCs versus the nodal TCs when the root-sum-of-squares bandwidth is calculated is

\[
\text{OCTC } \tau_{bw} \approx \sqrt{\tau_b^2 + \tau_c^2} = \sqrt{\tau_{se}^2 + (\tau_{cb} + \tau_{cc})^2} = \sqrt{(\tau_{se}^2 + \tau_{cb}^2 + \tau_{cc}^2) + 2 \cdot \tau_{cb} \cdot \tau_{cc}} \\
\text{Nodal TC } \tau_{bw} \approx \sqrt{\tau_b^2 + \tau_{cc}^2} = \sqrt{\tau_{se}^2 + \tau_{cb}^2 + \tau_{cc}^2} = \sqrt{(\tau_{se}^2 + \tau_{cb}^2 + \tau_{cc}^2) + 2 \cdot \tau_{cb} \cdot \tau_{cc}}
\]

The $\tau_{bw}$ differ in the last term under the radical in that $\tau_{cc} \neq \tau_c$. The nodal TC bandwidth is only accurate when $\tau_{cc} \approx \tau_c$. Equal bandwidth results only when

$$C_c = \frac{R_{se}}{R_I} \cdot C_e$$

The resulting $C_c$ value for many BJT circuits happens to also be in the range of the right-side expression, which might hide the above discrepancy from being experimentally detected. While the nodal-TC bandwidth happens to be approximately correct much of the time, it is not theoretically correct and is thus not necessarily even approximately correct relative to the OCTC basis for bandwidth.

Single-Stage BJT Single-Pole Approximation

Base-node bandwidth approximation can be placed in another perspective. The base-node resistance,
is related to the $R_{be}$ OCTC resistance by expressing the open-circuit base-emitter resistance,

$$R_{be} = r_e \parallel \frac{R_B + R_E}{1 + \frac{R_E}{r_m}}.$$

Then combining equations,

$$\frac{1}{r_m} - \frac{R_B}{R_E \cdot (r_e + R_E)} - \frac{R_{be}}{(R_B + R_E) \cdot r_e} \Rightarrow R_{be} = \frac{R_B \cdot (r_e + R_E)}{(R_B + R_E) \cdot r_e} \cdot R_{be}$$

Multiplying the rational factor by $R_y/R_b$, and applying the parallel-resistance formula, this reduces to

$$R_y = \frac{r_e \parallel R_E}{r_e \parallel R_E}, \quad R_{be} \Rightarrow R_{be} = \frac{r_e \parallel R_E}{R_B \parallel R_E} \cdot R_y$$

For the textbook CE case of $R_b = 0 \, \Omega$, then $R_y = R_{be}$. For the more common case of $R_b >> r_e$ and large $\beta_0$, $R_y = R_b$ and

$$R_{be} = r_e \cdot R_E >> r_e, \beta_0 \rightarrow \infty$$

Then the OCTC of $C_e$ is $\tau_e = R_{be} C_e = r_e \cdot C_e = \alpha_0 \cdot \tau_T \gg \tau_T$. Thus $\tau_e$ in fast amplifiers, with $R_b >> r_e$ and small $R_{be}$, is near $\tau_T$. Not uncommonly, $\tau_e < \tau_{cc}$, and to retain $\tau_e$ while omitting $\tau_{cc}$ makes a less accurate approximation.

By relating $R_{be}$ to $R_y$, both terms in the base-node time constant $\tau_b$ can be expressed in $R_y$ as

$$\tau_b = R_y \left[ \frac{r_e \parallel R_E}{R_B \parallel R_E} \cdot C_e + (1 + K_e) \cdot C_e \right] - R_b \cdot C_b$$

The result is a single equivalent capacitance at the base node of

$$C_b = \frac{r_e \parallel R_E}{R_B \parallel R_E} \cdot C_e + (1 + K_e) \cdot C_e$$

By applying $\alpha_0 \cdot \tau_T = r_e \cdot C_e$, $C_b$ becomes

$$C_b = \left( \frac{\alpha_0 \cdot \tau_T \cdot R_E}{(R_B \parallel R_E) \cdot (r_e + R_E)} \right) + (1 + K_e) \cdot C_e = \left( \frac{\alpha_0 \cdot \tau_T}{r_e + R_E} \right) + \left( \frac{R_B + R_E}{R_E} \right) + (1 + K_e) \cdot C_e$$

This is the exact expression for $C_b$, but it does not include $\tau_{cc}$.

The validity conditions of the $R_y \cdot C_b$ single-pole approximation can be found from the pole polynomial. It can be expressed in the above time constants as

$$s^2 \cdot (\tau_e \cdot \tau_{cc}) + s \cdot (\tau_e + [\tau_{cc} + \tau_{cb}]) + 1$$

The Miller effect at the base contributes only to $\tau_{cb}$ in the linear coefficient, and as it increases, the damping, $\zeta$ increases without any effect on pole magnitude, $1/\tau_{tr}$, which is set by the two time constants in

$$\tau_n = \sqrt{\tau_e \cdot \tau_{cc}}$$
For $\tau_{cb} = 0$ s, the polynomial has LHP real roots at $\tau_e$ and $\tau_{cc}$, both of which are often much less than $\tau_{cb}$. Then $\tau_n$ is small relative to the linear coefficient, the pole-pair is overdamped and the poles are real and separated. In this case, $\tau_n$ of the quadratic term of the polynomial is much less than the linear term and can be discarded. Put another way, when $\tau_e$ and $\tau_{cc}$, taken together, are much less than $\tau_{be}$, a single-pole approximation is valid - but which approximation?

The Miller term from $\tau_{cb}$ in $C_b$ dominates and the $\tau_e$ term only improves the approximation somewhat while $\tau_{cc}$ is discarded. Whereas the single-pole approximation based on the base node is valid for a dominant $\tau_{cb}$, a better single-pole approximation would include $\tau_{cc}$, for it is not uncommonly about the same in value as $\tau_e$ or greater for large $R_L$ and small $R_E$ (that makes $\tau_e$ large). The better single-pole approximation for the input-loop time constant is the full linear coefficient, the sum of the OCTCs that includes $\tau_{cc}$:

$$\text{OCTC } \tau_i \approx \tau_e + \tau_{cc} + \tau_{cb} \gg \tau_e + \tau_{cc}$$

Only when $\tau_{cc} \ll \tau_e$ and $\tau_{cb}$ does $R_bC_b$ offer an approximation of comparable accuracy. And better yet is the OCTC (root-sum-of-squares) bandwidth formula itself.