Iterated-map circuit creates chaos

Lars Keuninckx - July 13, 2016

The Design Idea circuit shown below is a simple implementation of an iterated unimodal map, reminiscent of the logistic or Verhulst map encountered in the study of nonlinear dynamics. It is useful to show chaotic discrete-time dynamics to students, or as a random number generator. Specifically, the circuit implements: $V_{k+1} = rF(V_k)$, where $F$ is a nonlinear unimodal function (a "bump"), implemented by the circuit in the dashed box. The response of this circuit is shown in the $V_{\text{out}}$ vs. $V_{\text{in}}$ plot.

![Graph showing response of the Q1 unimodal function circuit](image)

**Figure 1**  Response of the Q1 unimodal function circuit

$r$ is a gain factor provided by U1b. U1a and U3a are simply buffers. During the low phase of the clock signal, the output voltage of the nonlinear function is stored on C1. When the clock switches to high, this voltage is transferred to C2, and then used as the next input to the unimodal function. The supply voltage is 12V. S3 (¼ 4066) is used as an inverter.
For low gain settings ($r$), the iterates have the origin, zero volts, as stable equilibrium: $V^* = rF(V)$. By increasing the gain, the circuit moves from a stable equilibrium to $n$-periodic oscillations. The iterates of an $n$-periodic oscillation can be seen as equilibria of the $n$-iterated map, e.g., $V^* = rF(rF(rF(V)))$ for 3-periodic oscillations. This is shown in the first oscilloscope screenshot, for $r=4.7$. 

**Figure 2** Schematic
Figure 3 Behaviour for $r=4.7$ (red trace: $V_k$, blue trace: clock).

Further increasing the gain leads to chaos. Each initial condition or starting voltage in theory leads to a never repeating sequence of iterates. The logistic map $x_{n+1} = rx_n(1-x_n)$ is the best known model for such an iterated chaotic system, and uses a parabolic function. However, it turns out that any unimodal map, including discontinuous maps such as the mod-map, or non-smooth maps such as the tent map, also lead to chaos, hence the use of the simple circuit around transistor Q1.
Figure 4  Behaviour for $r=6.7$, made by setting the scope persistence to infinity and triggering as close as possible to the top of the voltages obtained.

Figure 4 shows a key property of deterministic chaos: sensitive dependence on initial conditions. Nearby (but slightly unequal) initial conditions (trigger voltages) lead to diverging iterate series after a just a few updates. Loosely speaking, there is a kind of “event horizon”. It is easy to predict the next iterate, but impossible to predict, say, the twentieth, from the current state, because we then have to know the initial condition with very high precision. It is important to note that even if this system could be made noiseless (as for the theoretical logistic map), it would output a chaotic sequence.

As for applications, the circuit can be used as a random number generator, real-life demonstrator of chaos, or just to flabbergast the local lab smartie who claims they’ve seen every circuit in the book.

—Lars Keuninckx spent over ten years in industry, designing automotive, industrial, and medical electronics, before returning to college to work on a physics degree. His interests include applications of nonlinear dynamics in electronic circuits, chaos, and neural networks. He recently obtained a PhD in engineering at the Applied Physics Research Group (APHY) of the Vrije Universiteit Brussel on the applications of delay systems and reservoir computing.

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